Thermodynamical, geometrical and Poincaré methods for charged black holes in presence of quintessence

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Abstract

Properties pertaining to thermodynamical local stability of Reissner-Nordström black holes surrounded by quintessence as well as adiabatic invariance, adiabatic charging and a generalized Smarr formula are discussed. Limits for the entropy, temperature and electric potential ensuring stability of canonical ensembles are determined by the classical thermodynamical and Poincaré methods. By the latter approach we show that microcanonical ensembles (isolated black holes) are stable. Two geometrical approaches lead to determine the same states corresponding to second order phase transitions.

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1 Introduction

Quintessence has made the subject of many papers ranging from exact solutions [1] to cosmological models [2] (and references therein) all fueled by observational data, which were achieved through various projects [3], and the discovery of the acceleration of the universe. These observation lead to believe that the accelerated expansion is due an exotic fluid with negative pressure making up the quintessence.

The spherically symmetric static solutions derived in [1], with quintessence as source term, obey a special condition of additivity and linearity in the energy-momentum tensor. They do not exhaust the set of spherically symmetric static solutions and the quest for new solutions remains open.

The solutions derived in [1] depend on four parameters: the mass and charge (M, q) of the Reissner-Nordström (RN) black hole and (c, ω) where $c > 0$, which might be called the charge of quintessence, determines the energy density ρ_q of quintessence and $-1 < \omega < 0$ relates pressure to energy density

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via the equation of state $p_q = \omega\rho_q$. A first classification of these solutions is based on their asymptotic behavor:

$$-1/3 \leq \omega < 1: \text{ asymptotically flat solutions}$$
$$-1 < \omega < -1/3: \text{ non-asymptotically flat solutions}.$$

Only the physical, geometrical and thermodynamical properties of the non-asymptotically flat solution with $\omega = -2/3$ were investigated in [1]. One of the aims of this paper is to discuss those physical and thermodynamical properties of asymptotically flat solutions ($-1/3 \leq \omega < 1$) pertaining to thermodynamical stability (Sections 2 and 3). We mainly discuss: horizon location, extremality conditions, relevant thermodynamic entities, generalized Smarr formula and first law of thermodynamics, adiabatic invariance and adiabatic charging of RN black holes with or without quintessence. Non-asymptotically flat solutions will make the subject of a subsequent work.

Gravitating systems do not obey the linear rules for mass and entropy additions. Thermodynamical methods appealing to ensembles where mass or entropy is held constant (and used as a control parameter) do not apply to gravitating systems [4–6]. An instance of that, the thermodynamic stability analysis or phase transitions of an isolated black hole (mass does not fluctuate) can't be carried out by either classical thermodynamical or geometrical methods which rely on the linearity hypothesis.

Poincaré [7] developed a powerful method applicable to problems pertaining to equilibrium and conditions of stability. Originally the method, known as the turning point method (TPM), was applied to the uniform rotational motion of a homogeneous liquid to determine the cases of local equilibrium and the conditions of stability of such equilibrium. Then it was applied to different situations [8]- [13] including problems non-tractable by classical thermodynamical or geometrical methods. It was generalized to many-parameter equilibrium families [14].

It is always possible to subdivide the space of all equilibrium configurations into 1-parameter subspaces. In each subspace, all points representing equilibrium states are related by varying one parameter called the control parameter; hence the name of linear series of equilibrium given to each subspace. The method of Poincaré [7] consists in further subdividing each linear series of equilibrium into smaller subspaces labeled stable, less stable, ..., unstable states of equilibria. The method employs the terminology of "degree of stability", increasing or decreasing depending on the number of negative modes of the associated Hessian matrix, and no notion of phase transition is employed. The choice of the Massieu function or the thermodynamic potential depends on the thermodynamic ensemble under consideration and in many cases there is no need to evaluate the eigenvalues of the Hessian matrix associated with the Massieu function or the thermodynamic potential [8]- [12].

Section 4 is devoted to the determination of the conditions of stability of equilibrium configurations of RN black holes surrounded by quintessence. We apply both, but in reversed order, the TPM and, whenever applicable, the classical thermodynamic approach. We will be able to always reach the same conclusion regarding the stability conditions but using different terminologies.

Geometrical methods [15]- [19] are geometric approaches which attach a measure of length to the

space of all equilibrium configurations, the metrics of which are built up from a Legendre-invariant thermodynamic potential and its first and second order partial derivatives with respect to a set of extensive variables. It is well known that stability analysis results depend on the thermodynamic ensemble [4–6, 8, 10], despite this fact the results derived in [15]- [18] do not make reference to any ensemble. Thermodynamic ensembles were used for the first time in [19]. The geometric methods rely on the linearity hypothesis and so are not applicable to isolated self gravitating systems. In Section 5 we define, within the contexts of these geometric approaches, the canonical ensemble describing the thermodynamic of an RN black hole immersed in a heat bath then apply the methods to analyze its stability. We conclude in Section 6.

2 Properties of RN black holes in presence of quintessence

In 4-dimensional spacetime, a spherical symmetric RN black hole plunged into the field of a spherical symmetric quintessence has the metric [1]

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 - f^{-1}(r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{2.1}$$

with[1]

$$f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} - \frac{2c}{r^{3\omega+1}}\,,\ -1 < \omega < 0 \text{ and } c > 0\,. \tag{2.2}$$

With this notation, the density of energy of quintessence is

$$\rho_q = -\frac{3\omega c}{r^{3\omega+3}} > 0\,. \tag{2.3}$$

Note that the integration of ρ_q over a hyper-surface of constant t diverges as $\lim_{r\to\infty} r^{-3\omega}$. As we shall see later, this results in a similar divergence of the Komar mass of the black hole. This infinite background contribution due to the density of quintessence should be subtracted in an appropriate way so that the Komar mass of the hole remains finite.

The physical properties of RN black holes described by (2.1) depend on the sign of $3\omega+1$. We shall consider the case where the asymptotic behavior of the hole is not altered by the presence of quintessence, that is the case where the black hole solution is still asymptotically flat. This correspondences to $3\omega+1 \geq 0$ ($-1/3 \leq \omega < 0$) with further constraints as shown below.

In the case $3\omega+1 = 0$, the metric (2.1) is not always asymptotically flat. If we assume $M > 0$ then 1) if $1-2c > 0$ ($0 < c < 1/2$), the metric may be brought to the following form upon performing the coordinate and parameter transformations: $t' = \sqrt{1-2c}\,t$, $r' = r/\sqrt{1-2c}$, $M' = M/(1-2c)^{3/2}$ and $q' = q/(1-2c)$:

$$\mathrm{d}s^2 = f'\mathrm{d}t'^2 - f'^{-1}\mathrm{d}r'^2 - (1-2c)r'^2\mathrm{d}\Omega^2 \tag{2.4}$$

where $f' = 1 - 2M'/r' + q'^2/r'^2$. This asymptotically flat metric (2.4) has a conical singularity in each

[1] In the original derivation of (2.2), c was taken negative [1]. We have made the substitution $-c \to 2c$ for simplicity and convenience.

plane $\theta = \theta_0 =$ constant. The deficit angle depends on θ_0 and is equal to $4\pi c$ in the plane $\theta = \pi/2$. 2) if $1-2c \leq 0$, the metric (2.1) is no longer asymptotically flat.

Let $-1/3 < \omega < 0$ $(1 > 3\omega+1 > 0)$. The horizons are defined by the condition $f(r) = 0$. Setting $u = 1/r$, this implies

$$1 - 2Mu + q^2u^2 = 2cu^{3\omega+1}. \tag{2.5}$$

The parabola $y = 1-2Mu+q^2u^2$ has an absolute minimum value of $(q^2-M^2)/q^2$ at $u_{\min} = M/q^2$. Thus, if $q^2 \leq M^2$ the graphs of $y = 1-2Mu+q^2u^2$ and $y = 2cu^{3\omega+1}$ always intersect, as shown in Figure 1 (a), at two points (u_h, u_1) such that $u_h < u_+$ and $u_1 > u_-$ with $1/u_+ = r_+ \equiv M + \sqrt{M^2-q^2}$ and $1/u_- = r_- \equiv M - \sqrt{M^2-q^2}$. The event horizon is at $r_h = 1/u_h > r_+$. Here is another effect of quintessence: For fixed M and q, the radius of the event horizon increases leading to higher entropy with respect to that of an RN black hole.

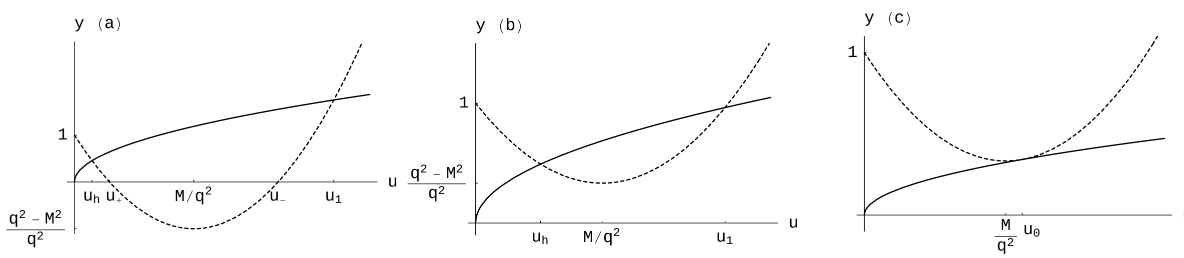

Figure 1: Plots of $y = 1-2Mu+q^2u^2$ (dotted line) and $y = 2cu^{3\omega+1}$ for $-1/3 < \omega < 0$. (a) A black hole: $q^2 \leq M^2$. The two curves intersect at two points labeled (u_h, u_1). (b) A black hole: $q^2 > M^2$ and $c > c_c$. The two curves intersect at two points labeled (u_h, u_1). (c) An extreme black hole: $q^2 > M^2$ and $c = c_c$. The two curves intersect at the point u_0 given by (2.9).

Even in the case $q^2 > M^2$, the two curves $y = 1-2Mu+q^2u^2$ and $y = 2cu^{3\omega+1}$ may still meet at two points (u_h, u_1) as shown in Figure 1 (b). Hence, the presence of quintessence makes it possible for the black hole to absorb more charge than the ordinary RN black hole before it becomes a naked singularity. For fixed ω, M and $q^2/M^2 > 1$ there is a critical value c_c of c below which the two curves do not intersect. c_c is such that the two curves $y = 1-2Mu+q^2u^2$ and $y = 2cu^{3\omega+1}$ have a common tangent line at the unique point of intersection $u_0 > 0$:

$$1 - 2Mu_0 + q^2{u_0}^2 = 2c_c{u_0}^{3\omega+1} \tag{2.6}$$

$$-M + q^2u_0 = c_c(3\omega+1){u_0}^{3\omega}. \tag{2.7}$$

Eliminating $c_c{u_0}^{3\omega}$ leads to

$$(1-3\omega)q^2{u_0}^2 + 6M\omega u_0 - (3\omega+1) = 0 \tag{2.8}$$

which has only one positive root given by

$$u_0 = \frac{\sqrt{9\omega^2(M^2-q^2)+q^2} - 3\omega M}{(1-3\omega)q^2} = \frac{\sqrt{9\omega^2M^2+(1-9\omega^2)q^2} - 3\omega M}{(1-3\omega)q^2}. \tag{2.9}$$

Using this in (2.7) we obtain

$$c_c = \frac{q^2 u_0 - M}{(3\omega+1){u_0}^{3\omega}} \tag{2.10}$$

for an extreme black hole solution. Since the r.h.s of (2.7) is positive, we must have $u_0 > M/q^2$, which is directly derived from (2.9). The roots (u_h, u_1) are such that

$$u_h < u_0 < u_1 \text{ and } M/q^2 < u_1 \quad (\text{for } c > c_c) \tag{2.11}$$

but u_h may be greater or smaller than M/q^2.

For fixed $(\omega, M, q^2/M^2 > 1)$ we conclude that if $c < c_c$, the metric (2.1) is a naked singularity and if $c > c_c$, the solution is a black hole with an event horizon at r_h and an inner horizon at r_1. The case $c = c_c$ corresponds to an extreme black hole with an event horizon at $r_0 = 1/u_0$ with u_0 given by (2.9).

We would formulate the last statement in terms of fixed (ω, M, c) if it were possible to solve (2.10) for q^2/M^2. An extreme black hole is obtained a) upon fixing $(\omega, M, q^2/M^2 > 1)$ and decreasing the value of c till $c = c_c$, as done in Figure 1, or b) upon fixing (ω, M, c) and increasing q^2/M^2 till $q^2/M^2 = (q^2/M^2)_c$. The case b) is generally not possible because we cannot invert (2.10), however, for fixed $c \ll 1$ and given (ω, M) the extreme RN black hole in presence of quintessence is approximated by

$$\Big(\frac{q^2}{M^2}\Big)_c = 1 + \frac{c}{M^{3\omega+1}} + \mathscr{O}\Big(\frac{c}{M^{3\omega+1}}\Big)^2 \qquad (c \ll 1)\,. \tag{2.12}$$

Recall that $1 > 3\omega + 1 > 0$, so that the contribution of the second term is not neglected even for relatively large black holes. For $3\omega + 1 = 0$ ($\omega = -1/3$) and $0 < c < 1/2$, it is easy to show using (2.4) or (2.1, 2.2) that the solution is a black hole if $q^2/M^2 < 1/(1-2c)$ and that the extreme black hole corresponds to

$$\Big(\frac{q^2}{M^2}\Big)_c = \frac{1}{1-2c} \quad (0 < 1-2c < 1)\,. \tag{2.13}$$

3 Thermodynamic of RN black holes with quintessence

In this section we only consider asymptotically flat black hole solutions which correspond to

$$[(q^2 \leq M^2 \text{ and any } c > 0) \text{ or } (q^2 > M^2 \text{ and } c > c_c)] \text{ if } -1/3 < \omega < 0 \tag{3.1}$$

$$\Big[\Big(q^2 \leq M^2 \text{ and } \frac{1}{2} > c > 0\Big) \text{ or } \Big(q^2 > M^2 \text{ and } \frac{1}{2} > c > \frac{q^2 - M^2}{2q^2}\Big)\Big] \text{ if } \omega = -1/3\,. \tag{3.2}$$

We first consider the case $-1/3 < \omega < 0$. We may take as thermodynamic state variables of these black holes the entropy S, the electric charge q and the parameter c. The entropy of such black holes is usually taken as $S = \pi {r_h}^2$, which is the area of the horizon by 4; however, for simplicity of notation we will work with the quantity $s = S/\pi$. Using $f(r_h) = 0$, we express the (gravitational) mass M in terms of (s, q, c) as follows

$$M = \frac{s + q^2 - 2cs^{W_-}}{2\sqrt{s}} \qquad \Big(W_\pm = \frac{1}{2} \pm \frac{3\omega}{2}\Big)\,. \tag{3.3}$$

It is straightforward to show that, in presence of quintessence, M is still the (internal) energy of the black hole. For that purpose we need to show that $(\partial M/\partial s)_{q,c}$ is indeed the temperature of the black hole. The latter is proportional to the surface gravity $T = \kappa/2$ and $\kappa = \partial_r f(r)/2$. Using these equations we express T is terms of the entropy and mass as

$$T = \frac{1}{2}\Big[\frac{M}{s} - \frac{q^2}{s^{3/2}} + \frac{(3\omega+1)c}{s^{(3\omega+2)/2}}\Big]. \tag{3.4}$$

Replacing M by the r.h.s of (3.3) reduces the r.h.s of (3.4) to $(\partial M/\partial s)_{q,c}$.

It is worth mentioning that the Komar mass, $M_{\text{Komar}} = \lim_{r\to\infty} r^2 \partial_r f/2$, diverges as

$$M_{\text{Komar}} = \lim_{r\to\infty}\Big[M - \frac{q^2}{r} + \frac{(3\omega+1)c}{r^{3\omega}}\Big].$$

However, if one subtracted the infinite background contribution due to quintessence, the remaining contribution approaches M.

Let (A_0, A_Q) be the functions

$$A_0 = \Big(\frac{\partial M}{\partial q}\Big)_{S,c},\; A_Q = \Big(\frac{\partial M}{\partial c}\Big)_{S,q} \;\text{ along with }\; T = \Big(\frac{\partial M}{\partial s}\Big)_{q,c}.$$

Using (3.3), we obtain

$$A_0 = \frac{q}{\sqrt{s}}, \quad A_Q = -\frac{1}{s^{3\omega/2}}, \tag{3.5}$$

$$T = \frac{s - q^2 + 6c\omega s^{W_-}}{4s^{3/2}}. \tag{3.6}$$

Using $1/s = {u_h}^2$ in (3.4), we see that $T \propto [c(3\omega+1){u_h}^{3\omega} + M - q^2 u_h]$. For the extreme black hole, $c = c_c$ and $u_h = u_0$, so by (2.7), $T = 0$.

Note that the r.h.s of (3.3) is a homogenous function of $(s, q^2, c^{2/(3\omega+1)})$ of degree $1/2$:

$$M(\lambda s, \lambda q^2, \lambda c^{2/(3\omega+1)}) = \lambda^{1/2} M(s, q^2, c^{2/(3\omega+1)}).$$

By Euler's theorem we obtain the generalized Smarr formula

$$M = 2Ts + A_0 q + (3\omega+1)A_Q c \tag{3.7}$$

where A_0 is the electric potential and A_Q is the potential associated with quintessence. It is straightforward to check that changes in the thermodynamic state variables (s, q, c) by amounts $(\mathrm{d}s, \mathrm{d}q, \mathrm{d}c)$ result in

$$\mathrm{d}M = T\mathrm{d}s + A_0 \mathrm{d}q + A_Q \mathrm{d}c \tag{3.8}$$

which is the first law of black-hole thermodynamics for RN black holes in a background of quintessence.

For the case $\omega = -1/3$ we use (2.4) where the hole has mass M' and charge q'. The horizon area is multiplied by $1-2c$ so that the entropy divided by π is $s' = (1-2c)(M' + \sqrt{M'^2 - q'^2})^2$. Let $s \equiv s'/(1-2c) = (M' + \sqrt{M'^2 - q'^2})^2$, then

$$M' = \frac{s + q'^2}{2\sqrt{s}} \quad \text{and} \quad T' = \frac{s - q'^2}{4(1-2c)s^{3/2}} \tag{3.9}$$

[where $T' \equiv (\partial M'/\partial s')_{q',c}$]. With $A'_0 = (\partial M'/\partial q')_{S',c}$, the generalized Smarr formula for this case takes the form

$$M' = 2T's' + A'_0 q' \,. \tag{3.10}$$

Particle absorption–emission: Adiabatic invariance. Without loss of generality, we assume $q > 0$. As shown in [20], a particle of mass m and electric charge $\varepsilon > 0$ moving in the geometry described by (2.1, 2.2) has the conserved energy for radial motion

$$E = m\sqrt{f(r) + (\mathrm{d}r/\mathrm{d}\tau)^2} + \frac{\varepsilon q}{r} \tag{3.11}$$

where τ is the proper time. If the particle crosses the horizon r_h, this incurs changes in (M, q): $\mathrm{d}M = E$, $\mathrm{d}q = \varepsilon$, which are related by (3.8) [here we assume that the motion of the particle does not affect the density of quintessence, that is we take $\mathrm{d}c = 0$]

$$2\left(\mathrm{d}M - \frac{q\mathrm{d}q}{r_h}\right) = [1 - q^2 {u_h}^2 + 6c\omega {u_h}^{3\omega+1}]\mathrm{d}r_h \,. \tag{3.12}$$

Using $f(r_h) = 0$, we bring it to the form

$$\mathrm{d}M - \frac{q\mathrm{d}q}{r_h} = -[W_- q^2 {u_h}^2 + 3M\omega u_h - W_+]\mathrm{d}r_h \tag{3.13}$$

where the expression inside the square parentheses is proportional to the l.h.s of (2.8). Since, by (2.11), $u_h < u_0$, the coefficient of $\mathrm{d}r_h$ in (3.13) is positive. Thus

$$\mathrm{sgn}\left(\mathrm{d}M - \frac{q\mathrm{d}q}{r_h}\right) = \mathrm{sgn}\,(\mathrm{d}r_h) \geq 0 \;\; \text{(by the second law)} \tag{3.14}$$

which we rewrite for the charged particle

$$\mathrm{sgn}\left(E - \frac{q\varepsilon}{r_h}\right) = \mathrm{sgn}\,(\mathrm{d}r_h) \geq 0 \;\; \text{(by the second law)}\,. \tag{3.15}$$

Hence, if the particle's energy is $E_0 \equiv q\varepsilon/r_h$, the latter reaches the horizon with a zero speed: $\mathrm{d}r/\mathrm{d}\tau = 0$ (as seen from (3.11)). In this case the particle is adiabatically accreted by the hole causing no change in the horizon's area: $\mathrm{d}r_h = 0$ (by (3.15)).

If $E > E_0$ the particle plunges into the hole with some kinetic energy causing the horizon to expand

by the amount

$$(E-E_0)/[W_+ - 3M\omega u_h - W_- q^2 {u_h}^2]$$

[particles with energy $E<E_0$ cannot reach the horizon].

Conversely, as shown in [21], black holes with relatively high temperature radiate electrically charged particles in the superradiant regime. The strong gravitational field near the horizon creates two particles of opposite charges. The total energy lost by the field is greater than the energy carried by each particles and in any case[2] $\mathrm{d}M<0$. If $\mathrm{d}q>0$, that is the particle with positive charge falls into the hole and the particle with negative charge escapes to infinity, the l.h.s of (3.14) is negative (since $\mathrm{d}M<0$) leading to $\mathrm{d}r_h<0$ which is not favored by the second law. Thus, by the second law of thermodynamics, the black hole may radiate charges of only the same sign as its own charge q reducing its charge upon receiving electric charges of opposite sign to its charge so that $\mathrm{d}q<0$. The process may continue till the entropy reaches its maximum values or proceed adiabatically (i.e. the charge ε of the falling particle is such that $\mathrm{d}M=q\varepsilon/r_h<0$) till the temperature drops.

Adiabatic charging of an RN black hole. We consider an ordinary RN black hole (BHn), where $c=0$, along with another RN black hole surrounded by quintessence (BHq) for given and fixed values of (ω,c) with $-1/3<\omega<0$. Each hole has mass M_0 and charge q_0 satisfying $M_0/q_0>1$ as in Figure 1 (a). The horizons, r_+ of (BHn) and r_h of (BHq), are such that $u_h<u_+$.

Let r_e denotes r_+ or r_h ($u_e=1/r_e$ denotes $u_+=1/r_+$ or $u_h=1/r_h$). If the two black holes are charged adiabatically, r_e remains constant (throughout the rest of this section u_+ and u_h are then taken as constants), so that by (3.12), $\mathrm{d}M=q\mathrm{d}q/r_e$ leading to

$$M=\frac{u_e}{2}q^2+D \qquad (D\equiv M_0-\frac{u_e}{2}{q_0}^2) \tag{3.16}$$

where the constant $D>0$ since $u_e<M_0/{q_0}^2$ as shown in Figure 1 (a). Using (2.5) it is easy to show

$$\text{For (BHn): } 2Du_+=1 \qquad (2D=r_+) \tag{3.17}$$

$$\text{For (BHq): } 2Du_h=1-2c{u_h}^{3\omega+1}<1\,. \tag{3.18}$$

While the two holes are being charged (at the same rate), the point M/q^2 (Figure 1 (a)) moves progressively to the left since

$$\frac{M}{q^2}=\frac{u_e}{2}+\frac{D}{q^2}$$

is obviously a decreasing function of q^2. As M/q^2 meets first u_+, (BHn) turns into an extreme black hole and the charging process ends [since (3.14) is no longer valid] by cumulating the total charge and mass

$$\text{For (BHn): } u_+=\frac{u_+}{2}+\frac{D}{{q_t}^2}\Rightarrow q_t=r_+\,,\; M_t=r_+ \tag{3.19}$$

[2]Excluded is the case where both particles fall into the hole. If we neglect radiations par the particles, this corresponds to $\mathrm{d}M=0$, $\mathrm{d}q=0$ and $\mathrm{d}r_h=0$.

as expected (the subscript "t" for total). What happens at this moment to (BHq)? Let us look at the derivative of $M/q = u_e q/2 + D/q$:

$$\frac{\partial(M/q)}{\partial q} = \frac{u_e}{2} - \frac{D}{q^2}.$$

This is zero if $q^2 = 2D/u_e$. For (BHn), using (3.17) along with $u_e = u_+$, we obtain $q = r_+ = q_t$, which means that M/q reaches its minimum value, 1, at the end of the charging process. For (BHq) the situation is quit different: At the moment M/q^2 meets u_+, the function M/q is still decreasing, so that (BHq) cumulates more charge and mass than (BHn), and reaches its minimum value at the moment M/q^2 meets u_h with

$$\text{For (BHq): } (M/q)_{\min} = \sqrt{2Du_h} < 1. \tag{3.20}$$

As the point M/q^2 passes u_h, the ratio M/q starts to increase but remains smaller than 1. The charging process ends when (BHq) turns into an extreme black hole at the moment the point u_0 (Figure 1 (c)) meets u_h. The total charge and mass of (BHq) are then given by (use (2.8) and (3.16))

$$(1-3\omega){q_t}^2{u_h}^2 + 6M_t\omega u_h - (3\omega+1) = 0 \text{ and } 2M_t = u_h{q_t}^2 + 2D$$

leading to

$$\text{For (BHq): } q_t = r_h\sqrt{1+3\omega(1-Du_h)} < r_h \tag{3.21}$$

$$\text{For (BHq): } 2M_t = r_h[(1-3\omega)2Du_h + (3\omega+1)] < 2r_h\,. \tag{3.22}$$

It is easy to check that:

$$\text{For (BHq): } M_t/q_t < 1\,, r_+ < q_t < r_h \text{ and } r_+ < M_t < r_h\,. \tag{3.23}$$

4 Thermodynamic local stability

In this section we only consider thermodynamic processes for which c is held constant. The second law of (black holes) thermodynamics governs all the criteria for stability. These criteria, in form, depend on how we consider the system (here the black hole). We shall consider canonical and microcanonical ensembles (CE, ME, respectively). The CE will consist of the black hole in equilibrium with its thermal radiation, treated as a reservoir (heat bath) at constant temperature and the ME will be the case of a black hole isolated from its surroundings. It has been shown in many applications that the thermodynamic local stability of CEs may equally be treated by classical thermodynamic approaches based on the Hessian matrix of the entropy or, equivalently, of the energy [4–6] or by the TPM [8] derived by Poincaré [7]. Due to non-additivity of entropy and mass in general relativity [9], the classical thermodynamic approach does not apply to isolated black holes, for which we shall then apply the TPM.

4.1 CE: Classical thermodynamic approach

We assume that the hole is immersed in a thermal bath at constant temperature T. Applying the classical thermodynamic approach, we denote by $\mathrm{d}s_b$ the change in the entropy of the bath. Then any possible change in the state of the system requires $\mathrm{d}s+\mathrm{d}s_b \geq 0$. Conversely, if fluctuations try to take the system out of equilibrium with the reservoir, that is if

$$\mathrm{d}s+\mathrm{d}s_b < 0 \tag{4.1}$$

for allowed changes in the system's and reservoir's parameters, the system cannot leave the current state which is said to be in stable equilibrium with the bath. The inequality (4.1) is the condition from which all criteria for local stability of systems in contact with reservoirs are derived. Since any process is considered reversible for the (huge) reservoir, from (3.8) it follows that $\mathrm{d}s_b = (\mathrm{d}M_b - A_0\mathrm{d}q_b)/T$ ($\mathrm{d}c \equiv 0$). Exchanges between the system and the reservoir obey the conservation rules: $\mathrm{d}M_b = -\mathrm{d}M$ and $\mathrm{d}q_b = -\mathrm{d}q$. Using these equations in (4.1) leads to

$$\mathrm{d}M > T\mathrm{d}s + A_0\mathrm{d}q\,. \tag{4.2}$$

In (4.2) we are using the mass-energy as a fundamental thermodynamic quantity, instead of the entropy, because it is not possible to reverse (3.3) and express s in terms of M.

By the first law, $\mathrm{d}M = T\mathrm{d}s + A_0\mathrm{d}q$, the condition (4.2) is not sensitive to first order changes in the allowed parameters. Keeping up to second partial derivatives of M with respect to the extensive parameters (s,q) in the Taylor expansion of M, we obtain the equivalent condition (which may be written as a Hessian matrix)

$$\frac{1}{2}\left(\frac{\partial^2 M}{\partial s^2}\right)_{q,c}\mathrm{d}s^2 + \left(\frac{\partial^2 M}{\partial s\partial q}\right)_{c}\mathrm{d}s\mathrm{d}q + \frac{1}{2}\left(\frac{\partial^2 M}{\partial q^2}\right)_{S,c}\mathrm{d}q^2 > 0 \tag{4.3}$$

where all first order terms canceled out. With

$$\left(\frac{\partial^2 M}{\partial s^2}\right)_{q,c} = \frac{T}{C_q}\,,\ \left(\frac{\partial^2 M}{\partial s\partial q}\right)_{c} = \left(\frac{\partial T}{\partial q}\right)_{s,c}\,,\ \left(\frac{\partial^2 M}{\partial q^2}\right)_{s,c} = \left(\frac{\partial A_0}{\partial q}\right)_{s,c} \tag{4.4}$$

(T and A_0 have been defined in the equation preceding (3.5) and their expressions are given in (3.5, 3.7)), the condition (4.3) implies[3]

$$C_q > 0\,,\ \left(\frac{\partial A_0}{\partial q}\right)_{s,c} > 0 \text{ and } \left(\frac{\partial T}{\partial q}\right)^2_{s,c} < \frac{T}{C_q}\left(\frac{\partial A_0}{\partial q}\right)_{s,c}. \tag{4.5}$$

[3]We would like to show the analogy with classical thermodynamics via the following correspondences where (P,V) are the pressure and volume of the classical system: $\mathrm{d}q \to -\mathrm{d}V$, $A_0 \to P$ and $C_q \to C_V$. In a similar way, $(\partial q/\partial A_0)_{s,c}/q$, which may be called the factor of adiabatic charge, corresponds to the adiabatic compressibility $-(\partial V/\partial P)_S/V$. Similar terminology has been used in [22].

Here C_q is the specific heat at constant charge

$$\begin{aligned} C_q =& \Big(\frac{\partial M}{\partial T}\Big)_{q,c} = T/\Big(\frac{\partial^2 M}{\partial s^2}\Big)_{q,c} \\ =& \frac{2s(s-q^2+6c\omega s^{W_-})}{3q^2-s-6c\omega(2+3\omega)s^{W_-}}\,. \end{aligned} \tag{4.6}$$

Note that if T is constant (the case where the hole is immersed in a reservoir), the l.h.s of the third inequality in (4.5) is zero.

The mass M, given by (3.3), is supposed to be positive, so we have the extra condition

$$2\sqrt{s}M = s+q^2-2cs^{W_-} > 0\,. \tag{4.7}$$

Usually, in classical thermodynamics, all relevant thermodynamic quantities pertaining to the reservoir $(M_b, q_b, \cdots)$, but the temperature, are allowed to fluctuate. In this paper we shall allow T to fluctuate too and investigate separately the cases T constant and T fluctuating.

4.1.1 T constant

Condition (4.3) or (4.5) reduces to $C_q > 0$ and $T > 0$. Since $s > 0$ and $C_q \propto T$, we have to solve simultaneously

$$\text{(a): } s-q^2+6c\omega s^{W_-} > 0 \text{ and (b): } 3q^2-s-6c\omega(2+3\omega)s^{W_-} > 0\,. \tag{4.8}$$

For $c = 0$ we recover the conditions for ordinary RN black holes:

$$q^2 < s < 3q^2 \tag{4.9}$$

with $s = (M+\sqrt{M^2-q^2})^2$ leads to the known conditions for local stability [6]

$$3/4 < q^2/M^2 < 1\,. \tag{4.10}$$

Now back to quintessence case $c > 0$. For $-1/3 < \omega < 0$ we have $1/2 < W_- < 1$ and $1 < 2+3\omega < 2$. In order to solve the inequality (4.8) (a), we consider the line $y(s) = s-q^2$ and the concave-down curve $y(s) = -6c\omega s^{W_-}$ (the graph of which is similar to that of $y = \sqrt{s}$), it is easy to see that they do intersect at one and only one point $s_1(\omega) > q^2$. Thus $T > 0$ if $s > s_1(\omega)$, which solves (4.8) (a). Similarly the line $y(s) = s-3q^2$ and the concave-down curve $y(s) = -6c\omega(2+3\omega)s^{W_-}$ intersect at one and only one point $s_2(\omega) > 3q^2$. Thus (4.8) (b) is solved by $s < s_2(\omega)$. The black hole is locally stable if

$$s_1(\omega) < s < s_2(\omega) \text{ and } M > 0\,. \tag{4.11}$$

The first condition in (4.11) is a generalization of (4.9) where q^2 and $3q^2$ have been shifted to the right ($s_1(\omega) > q^2$, $s_2(\omega) > 3q^2$). The problem, however, cannot be solved as in (4.10) since (s_1, s, s_2) are not

known explicitly in terms of (M,q,ω,c).

In the case of ordinary RN black holes ($c=0$), the mass, $2\sqrt{s}M=(s+q^2)$, is a positive function of the entropy. In the case of RN black holes surrounded by quintessence, this is not guaranteed a priori and we need the solve the extra inequality (4.7). Consider the line $y(s)=s+q^2$ and the concave-down curve $y(s)=2cs^{W_-}$. For fixed (ω,c), if we choose $q^2>{q_c}^2\equiv W_+(2cW_-)^{1/W_+}/W_-$ [for ordinary RN black holes ($c=0$), ${q_c}^2=0$], the line and the curve do not intersect and $M>0$ for all s and if $q^2\leq {q_c}^2$, they meet at two points $s_4(\omega)\leq s_3(\omega)$ where $M>0$ for $s>s_3(\omega)$ or $s<s_4(\omega)$.

Figure 2 (a) shows plots of the curves $s_1\equiv s_1(\omega)$ (red), $s_2\equiv s_2(\omega)$ (blue) and $s_3\equiv s_3(\omega)$ (green) for $-1/3<\omega<0$. We have chosen $q=c=1$ in such a way that $s_4(\omega)<s_1(\omega)$ so that there is no need to plot the curve $s_4\equiv s_4(\omega)$. $T>0$ for $s>s_1$, $C_q>0$ for $s_1<s<s_2$ and $M>0$ for $s>s_3$. Three isotherm curves are shown in Figure 2 (a): $T=0$ (red), $T=1/(8\pi\sqrt{2})$ (brown) and $T=9/(4\pi 10^{3/2})$ (cyan). The plane region of local stability, identified by $T>0, C_q>0$ and $M>0$, is the region enclosed by the green curve, blue one and the s-axis for $\omega_0<\omega<0$ where $(\omega_0,s_0)=(-0.19,21.37)$ is the intersection point of the curves $s_2\equiv s_2(\omega)$ (blue) and $s_3\equiv s_3(\omega)$ (green). For q held constant, (ω_0,s_0) lies on the curve given by

$$s=3\frac{1-\omega(3\omega+2)}{1+3\omega(3\omega+2)}q^2\qquad(\forall c).$$

The region where the hole is unstable is above the blue curve for $\omega_0<\omega<0$. Since C_q diverges on the blue curve, this latter determines a limit for second order phase transition.

The conditions in (4.8) and (4.7) are rewritten as $T>0:\ q^2<s+6c\omega s^{W_-}$, $C_q>0:\ q^2>(s/3)+2c\omega(2+3\omega)s^{W_-}$ (with $T>0$) and $M>0:\ q^2>-s+2cs^{W_-}$, respectively. Figure 2 (b) shows plots of the surfaces $q^2=s+6c\omega s^{W_-}$ (red), $q^2=(s/3)+2c\omega(2+3\omega)s^{W_-}$ (blue) and $q^2=-s+2cs^{W_-}$ (green) for $c=1$. The curves $s_1\equiv s_1(\omega)$ (red), $s_2\equiv s_2(\omega)$ (blue) and $s_3\equiv s_3(\omega)$ (green) are projections of intersections of these surfaces with the plane $q^2=1$. The physical space region is all of the space region bounded above by the red surface and below by the green one if there $q^2>0$ or the subregion of it where $q^2>0$. If the point p representing the thermodynamic state of the black hole lies between the red and the blue surfaces and above the green one ($M>0$), the hole is thermodynamically stable against fluctuations in (s,q): fluctuations are "entropically" suppressed. Since T does not fluctuate, the stability condition is subject to the further constraint: r.h.s of (3.6) $=$ constant. For fixed c, this defines a new surface in the space (ω,s,q^2) of parameters. Stability concerns only those states p of the hole which lie on the segment of this new surface $T=$ constant which is sandwiched by the red and blue surfaces. The space region of instability is bounded above by the part of the blue surface where $q^2>0$ and below either by the green surface if there $q^2>0$ or by the plane $q^2=0$.

For $\omega=-1/3$ ($0<c<1/2$) we have seen that the solution is an ordinary RN black hole, given by (2.4), with new mass M' and charge q'. Using (3.9) we obtain

$$C_q'=(1-2c)\frac{2\pi s(s-q'^2)}{3q'^2-s}\,. \tag{4.12}$$

[Recall that in this case $s\equiv s'/(1-2c)=(M'+\sqrt{M'^2-q'^2})^2$]. Thus the conditions for local stability are

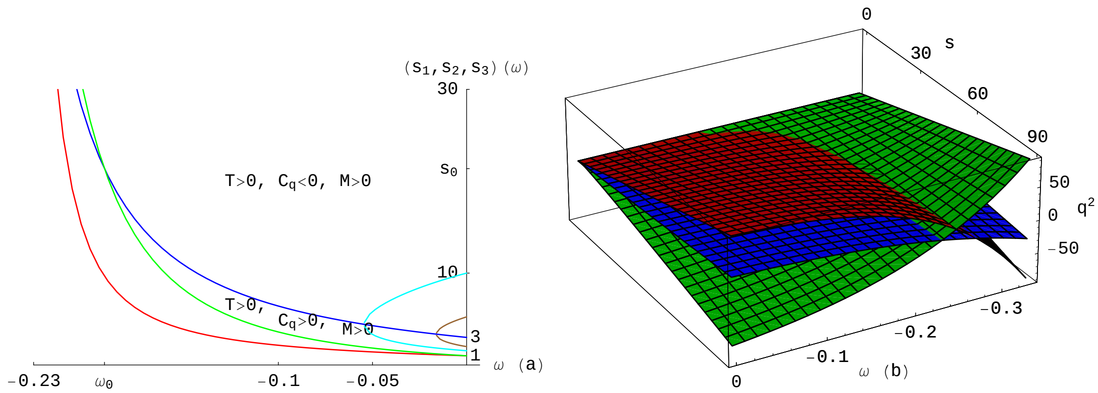

Figure 2: (a) Plots of $s_1 \equiv s_1(\omega)$ (red), $s_2 \equiv s_2(\omega)$ (blue) and $s_3 \equiv s_3(\omega)$ (green) for $-1/3 < \omega < 0$ and $q = c = 1$. $T > 0$ for $s > s_1$, $C_q > 0$ for $s_1 < s < s_2$ and $M > 0$ for $s > s_3$. The red, brown and cyan curves are the isotherms $T = 0$, $T = 1/(8\pi\sqrt{2})$ and $T = 9/(4\pi 10^{3/2})$. The curves $s_2 \equiv s_2(\omega)$ (blue) and $s_3 \equiv s_3(\omega)$ (green) intersect at the point $(\omega_0, s_0) = (-0.19, 21.37)$. The black hole surrounded by quintessence is locally stable in the region between the green and blue curves for $\omega_0 < \omega < 0$ and unstable in the region above the blue curve for $\omega_0 < \omega < 0$. Limited to these values of ω, the blue curve determines the points where C_q diverges, there the black hole undergoes a second order phase transition. (b) Plots of the surfaces $q^2 = s + 6c\omega s^{W_-}$ (red), $q^2 = (s/3) + 2c\omega(2+3\omega)s^{W_-}$ (blue) and $q^2 = -s + 2cs^{W_-}$ (green) for $c = 1$. $T > 0$ below the red surface, $C_q > 0$ below the red surface and above the blue one and $M > 0$ above the green one.

the same as in (4.9) but with M' and q', leading to

$$3/4 < (1-2c)q^2/M^2 < 1\,. \tag{4.13}$$

By (3.2) we see that black holes with either $q^2 \leq M^2$ or $q^2 > M^2$ are locally stable provided conditions (4.13) are satisfied. Black holes with $3M^2/4 > (1-2c)q^2$ are unstable for $C'_q < 0$. In the (M^2, q^2)-plane, the line $M^2 = 4(1-2c)q^2/3$ determines a phase transition of second order where C'_q diverges.

4.1.2 *T* fluctuating

In processes where T fluctuates, the conditions in (4.5) need be solved simultaneously. The second condition in (4.5) is already satisfied; using (3.5, 3.6, 4.6), the third one leads to

$$\frac{q^2 - s - 6c\omega(2+3\omega)s^{W_-}}{3q^2 - s - 6c\omega(2+3\omega)s^{W_-}} > 0\,. \tag{4.14}$$

Note that if the numerator of the fraction in (4.14) is positive, the denominator, which also appears in the expression of C_q, too will be positive. Thus, all we have to solve are the conditions (4.7, 4.8 (a)) and the new one $q^2 - s - 6c\omega(2+3\omega)s^{W_-} > 0$, all grouped into one system, respectively

$$s + q^2 > 2cs^{W_-} \ (M > 0) \tag{4.15}$$

$$s - q^2 > -6c\omega s^{W_-} \ (T > 0) \tag{4.16}$$

$$s - q^2 < -6c\omega(2+3\omega)s^{W_-}\,. \tag{4.17}$$

We first consider an ordinary RN black hole ($c = 0$). The last two conditions (4.16, 4.17) lead to $s = q^2$, then, by the first one, $s = q^2 > 0$. With $s = (M + \sqrt{M^2 - q^2})^2$, we obtain $M = q$. Thus, only the extreme RN black hole is stable against fluctuations in (s, q, T). The first law takes the simple form $\mathrm{d}M = \mathrm{d}q$: for any change in q there corresponds an equal change in M so that the hole remains extreme for any $\mathrm{d}q$.

In the generic case $c > 0$, $-1/3 < \omega < 0$ we have $1 < 2 + 3\omega < 2$. The solution of (4.17) is similar to that of (4.16): the line $y(s) = s - q^2$ intersects the concave-down curve $y(s) = -6c\omega(2+3\omega)s^{W_-}$ at one and only one point $s_5(\omega) > q^2$ [note that $s_5(\omega) < s_2(\omega)$]. Now, because $2 + 3\omega > 1$, the graph of $y(s) = -6c\omega(2+3\omega)s^{W_-}$ is above that of $y(s) = -6c\omega s^{W_-}$, the intersection of which with $y(s) = s - q^2$ determines the point $s_1(\omega) > q^2$ as discussed earlier. This leads to $s_1(\omega) < s_5(\omega)$. Thus, the three conditions (4.15 to 4.17) are solved by

$$s_1(\omega) < s < s_5(\omega) \;\text{ and }\; M > 0\,. \tag{4.18}$$

Allowing fluctuations in T, the condition for local stability has been narrowed since $s_5(\omega) < s_2(\omega)$ (compare with (4.11)). This is the region bounded below by the red curve and above by the magenta one of Figure 3 (a), which shows plots of the curves $s_1 \equiv s_1(\omega)$ (red), $s_2 \equiv s_2(\omega)$ (blue) and $s_5 \equiv s_5(\omega)$ (magenta) for $-1/3 < \omega < 0$, $c = 1$ and $q = 5$. The three curves do not intersect and admit the line $\omega = -1/3$ as vertical asymptote. For fixed (ω, q), as the entropy increases from the red line which defines the states of extreme black holes, the state of the hole crosses the magenta line and becomes unstable regarding fluctuations in T. However, this transition cannot be qualified a first order phase transition since the entropy is continuous there (no jump in s) and the phase of the hole is the same; rather one might qualify it a "behavioral" change or transition. In the region bounded below by the magenta curve and above by the blue one, the hole is, however, stable regarding changes in s, q, as we have seen in the previous subsection. So this is a phase transition from thermodynamic states, which are stable against fluctuations in (s, q, T), to states which are stable against fluctuations in (s, q) only. If the entropy continues to increase, the hole undergoes the above-mentioned second order phase transition by crossing the blue line where C_q diverges and changes the sign.

Figures 3 (b,c) show plots of the surfaces $q^2 = s + 6c\omega s^{W_-}$ (red), $q^2 = s + 6c\omega(2+3\omega)s^{W_-}$ (magenta), $q^2 = -s + 2cs^{W_-}$ (green) and the plane $q^2 = 25$ (yellow) for $c = 1$ [for clarity, the blue surface $q^2 = (s/3) + 2c\omega(2+3\omega)s^{W_-}$ is not shown]. The plane $q^2 = 25$ intersects the red surface, which represents the states of extreme black holes, along a curve the projection of which on the (ω, s)-plane is the red curve of Figure 3 (a). The other curves in Figure 3 (a) are also projections of intersections of the plane with the corresponding surfaces. For fixed (ω, s), as q^2 decreases, along a vertical line (Figure 3 (c)), from its value on the red surface to its value on the magenta one, the hole remains stable against fluctuations in T till the line crosses the magenta surface and the hole undergoes a behavioral change. As the charge decreases again, the state of the hole crosses the blue surface (not shown) and undergoes a second phase transition of second order.

We have thus shown that black holes with low entropy, or high charge or both are stable against fluc-

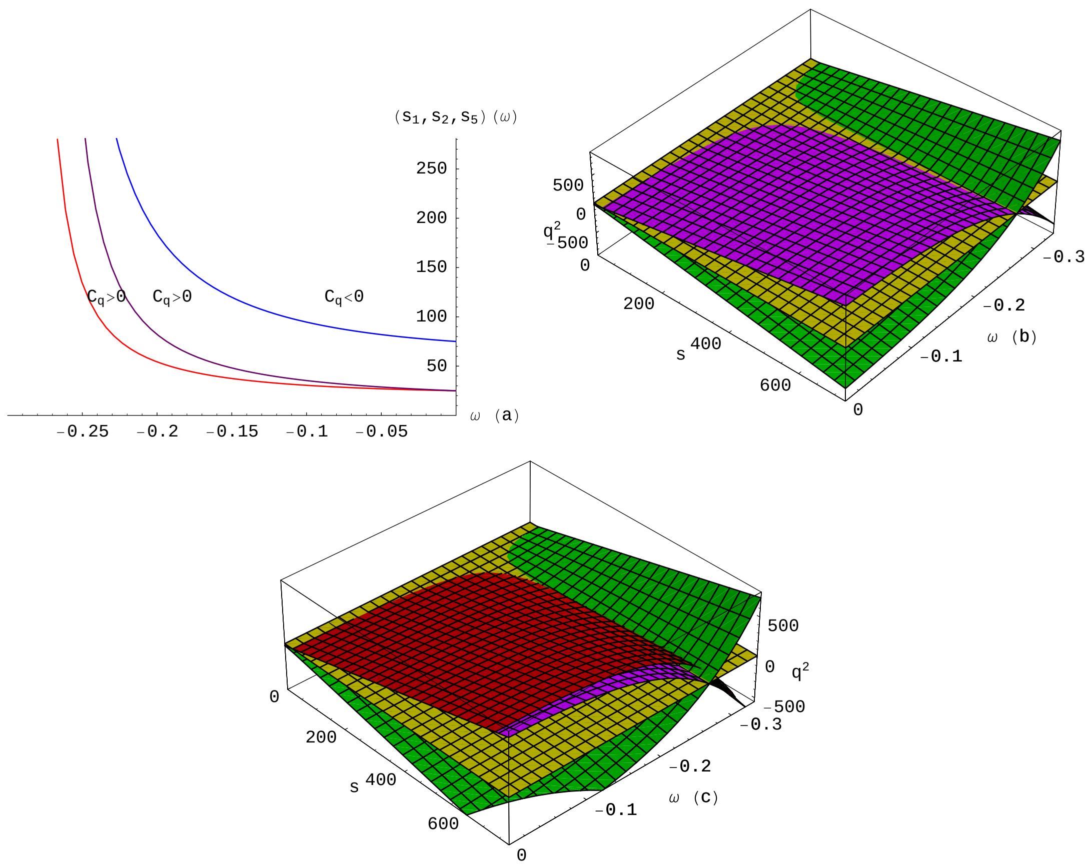

Figure 3: (a) Plots of $s_1 \equiv s_1(\omega)$ (red), $s_2 \equiv s_2(\omega)$ (blue) and $s_5 \equiv s_5(\omega)$ (magenta) for $-1/3 < \omega < 0$, $c = 1$ and $q = 5$. (b) The space region of stability against fluctuations in both (s, q, T) is above the magenta surface and (c) below the red one and where the green surface is below both of them.

tuations in (s, q, T), and black holes with relatively high entropy, or low charge or both are stable against fluctuations in (s, q) only. There is a behavioral change between the states of these holes maintaining the sign of $C_q > 0$. Finally, black holes with high entropy, or low charge or both are unstable against allowed fluctuations and have $C_q < 0$.

For $\omega = -1/3$ $(0 < c < 1/2)$, using Eqs. (3.11, 4.12) we bring the third condition in (4.5) to

$$\frac{q'^2 - s}{3q'^2 - s} > 0\,. \tag{4.19}$$

The problem is similar to that of ordinary RN black holes with M and q replaced by M' and q': Solving (4.19) together with $T' > 0$ and $C'_q > 0$ we obtain $M' = q'$ or $M = \sqrt{1-2c}\,q$. Only the extreme black hole is stable against fluctuations in (s, q). The first law takes the simple form $\mathrm{d}M = \sqrt{1-2c}\,\mathrm{d}q$: for any change in q there corresponds a proportional change in M so that the hole remains extreme for any $\mathrm{d}q$.

4.2 ME and CE by the TPM

We apply the TPM to the case of an isolated RN black hole surrounded by quintessence, which is the ME and extend the analysis to the case where the hole is immersed in a heat bath, which is the CE treated in the preceding subsection by the classical thermodynamic approach. The aim of applying the TPM to the CE is to rederive in an elegant way the results of subsubsections 4.1.1 and 4.1.2 using the powerful method of Poincaré [7].

In the TPM one uses the following Massieu functions Ψ: $(s, -F/T, -G/T, \cdots)$ or, equivalently, the thermodynamic potentials Ψ: $(M, F, G, \cdots)$ [8] depending on the ensemble, where $F = M - Ts$ and $G = F - A_0 q$ are the Helmholtz and Gibbs free energies, respectively. For our ensembles the corresponding potentials at equilibrium are:

$$\text{ME: } \Psi = M(s,q) \tag{4.20}$$

$$\text{CE: } \Psi = F(T,q) \text{ if } T \text{ constant} \tag{4.21}$$

$$\text{CE: } \Psi = G(T,A_0) \text{ if } T \text{ fluctuates}\,. \tag{4.22}$$

The TPM consists in plotting the planar curves $\partial\Psi/\partial x$ (all other variables kept constant) against x, which is some control parameter. The variable $\partial\Psi/\partial x$ is called the conjugate of x with respect to Ψ. These curves are called linear series of equilibrium. Changes of equilibrium (from stable to less stable to unstable and conversely) occur at points where the curves have vertical tangents or bifurcations. If the curve has no vertical tangents, as in Figure 4 (a), then all points on the curve have the same degree of stability: If it is known that a point on the curve represents a stable equilibrium state, then all the points on the curve represent similar states. If the curve, with vertical tangent, is concave left near the vertical tangent (as shown in Figure 4 (b,c)), then *all the points* on the branch of the curve where the slope is negative *near* the vertical tangent (upper branch in Figure 4 (b,c)) are more stable than the points on the other branch where the slope is positive *near* the vertical tangent[4] (lower branch in Figure 4 (b,c)).

For the ME, the control parameter is the entropy s. Since $(\partial\Psi/\partial s)_q = (\partial M/\partial s)_q = T$ we have plotted in Figure 4 (a) the series of equilibria $T(s)$, with q constant, for both the isolated RN black hole surrounded by quintessence (fitted line) and Schwarzschild black hole (dotted line). The curves approach each other as $s \to \infty$. Knowing that the isolated Schwarzschild black hole is locally stable [8], we conclude that the isolated RN black hole surrounded by quintessence is at least stable for large values of s. But since there is no change of stability on the series of equilibrium (no vertical tangents on the curve $y = T(s)$), the hole is then stable for all s.

The case of the CE has been split into tow subcase: T constant (subsubsection 4.1.1) and T fluctuating (subsubsection 4.1.2). In the former case, $\Psi = F(T,q)$ and T is a control parameter. Using the first law,

[4]The slope need not be of the same sign along a given branch. The rule is valid if Ψ is one of the potentials $(M, F, G\cdots)$, which are minimum for stable equilibria. For Ψ chosen from the list of Massieu functions $(s, -F/T, -G/T, \cdots)$, which are maximum for stable equilibria, then if the curve is concave left, all the points on on the branch of the curve where the slope is positive near the vertical tangent are more stable than the points on the other branch where the slope is negative near the vertical tangent.

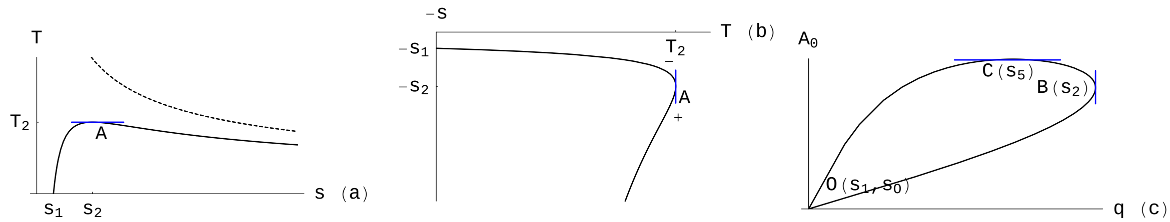

Figure 4: (a) $(q=1, c=1, \omega=-0.1)$: Plot of T vs. s at constant q for both the isolated (ME) RN black hole surrounded by quintessence (fitted line) and Schwarzschild black hole (dotted line). (b) $(q=1, c=1, \omega=-0.1)$: Plot of $-s$ vs. T at constant q for the RN black hole surrounded by quintessence immersed in a heat bath (CE). (c) $(T=0.05, c=1, \omega=-0.1)$: Plot of A_0 vs. q at constant T for the RN black hole surrounded by quintessence immersed in a heat bath (CE). It is a parametric plot for $s_1 \leq s \leq s_2$, $s_2 \leq s \leq s_O$, $s_1 = 0.329134$, $s_2 = 7.191$, $s_O = 14.65$, $q = \sqrt{s - 4Ts^{3/2} + 6c\omega s^{W_-}}$ and $A_0 = q/\sqrt{s}$.

$\mathrm{d}M = T\mathrm{d}s + A_0\mathrm{d}q$, we obtain

$$\mathrm{d}F = -s\mathrm{d}T + A_0\mathrm{d}q\,. \tag{4.23}$$

Hence, $(\partial\Psi/\partial T)_q = (\partial F/\partial T)_q = -s$. We need to plot the curve $-s(T)$ for q constant. This is the same curve plotted in Figure 4 (a) but rotated 90° clockwise as shown in Figure 4 (b). Notice that the point A where $-s(T)$ has a vertical tangent is the same point where $T(s)$ has a horizontal tangent. Using (3.6)

$$\Big(\frac{\partial T}{\partial s}\Big)_{q,c} = \frac{3q^2 - s - 6c\omega(2+3\omega)s^{W_-}}{8s^{5/2}} \tag{4.24}$$

we see that $(\partial T/\partial s)_{q,c} = 0$ at the points s_2 where C_q diverges. Thus we reach the same conclusion as in subsubsection 4.1.1: For fixed (q, c, ω), the stability breaks at the point A where there is a vertical tangent (where $C_q = \infty$), the points where $T < T_2 = T(s_2)$ and $s_1 < s < s_2$ are stable (negative slopes) and the points where $T < T_2$ and $s > s_2$ are unstable (positive slopes).

For the ordinary RN black hole $(c = 0)$, $s_1 = q^2$, $s_2 = 3q^2$ and $T_2 = 1/(6\sqrt{3}q)$. This is the maximum temperature beyond which the hole cannot resist evaporation.

Using (4.23), we have $(\partial\Psi/\partial q)_T = (\partial F/\partial q)_T = A_0$. So we could also plot $A_0(q)$ for T constant, using thus q as a control parameter. This is a parametric plot, shown in Figure 4 (c), parameterized by s. The entropy increases along the upper branch (stable) from s_1 at O to s_2 at B, it continues to increase along the lower branch (unstable) from B back to O. To show that, at B, $s = s_2$, which is the point where C_q diverges, we rewrite (3.6) as $q = \sqrt{s - 4Ts^{3/2} + 6c\omega s^{W_-}}$. With T constant ((c, ω) are also kept constant), at B we have $\mathrm{d}q/\mathrm{d}s = 0$ leading to $s - 6Ts^{3/2} + 6c\omega W_- s^{W_-} = 0$. Eliminating T by (3.6), the remaining equation is again $3q^2 - s - 6c\omega(2+3\omega)s^{W_-} = 0$ (compare with (4.24)), the solution of which is $s = s_2$. Here again we reach the same conclusion as in subsubsection 4.1.1.

If, in the CE, we assume that T fluctuates, the appropriate potential is the Gibbs function $\Psi = G(T, A_0)$ leading $\mathrm{d}G = -s\mathrm{d}T - q\mathrm{d}A_0$. Using A_0 as a control, we obtain $(\partial\Psi/\partial A_0)_T = (\partial G/\partial A_0)_T = -q$. So we need to plot the curve $-q(A_0)$ for T constant. This is the same curve plotted in Figure 4 (c) but rotated 90° clockwise. So the horizontal tangent at C becomes a vertical one and the (upper) branch from s_1 at O to s_5 at C is the stable one if fluctuations in T are taken into consideration [negative slopes near C

after revolving the curve in Figure 4 (c) 90° clockwise]. The branch from C through B back to O is the unstable one regarding fluctuations in T. So we just re-derived the conclusion made in subsubsection 4.1.2 (compare with (4.18)). All we need to show is $s = s_5$ at C. At C, $\mathrm{d}A_0/\mathrm{d}s = 0$ which is the same as $2Ts^{3/2} + 3c\omega(1+3\omega)s^{W_-} = 0$. Eliminating T by (3.6), the remaining equation is again $q^2 - s - 6c\omega(2+3\omega)s^{W_-} = 0$ (compare with (4.17)), the solution of which is $s = s_5$.

5 Phase transition via geometric methods

In this section we briefly present the results of thermodynamic stability as derived by the two geometric approaches: 1) Geometrothermodynamics (GTD) [15] and 2) Liu-Lu-Luo-Shao (LLLS) method [18].

5.1 Geometrothermodynamics

Let Ψ be a thermodynamic potential and (E^a, I^a) the set of associated extensive and intensive variables. We define a $(2n+1)$-dimensional space $\mathbb{T}$ whose coordinates is the set $Z^A = \{\Psi, E^a, I^a\}$ where $A : 0 \to 2n$ and $a : 1 \to n$. Together with the Gibbs 1-form $\Theta = \mathrm{d}\Psi - \delta_{ab} I^a \mathrm{d}E^b$, $(\mathbb{T}, \Theta)$ make up the $(2n+1)$-dimensional contact manifold of metric $G^{AB}(Z^C)$, which is the thermodynamic phase space [15]. Here we assume that $\mathbb{T}$ is differentiable and that Θ satisfies the condition $\Theta \wedge (\mathrm{d}\Theta)^n \neq 0$. The subspace $\mathbb{E} \subset \mathbb{T}$ of equilibrium states is defined by the map: $\varphi : \mathbb{E} \to \mathbb{T}$ such that the pullback vanishes at Θ: $\varphi^*(\Theta) \equiv 0$.

The singularities of the curvature scalar R_{GTD} of $\mathbb{E}$ determine the points or states where there are second order phase transitions of the thermodynamic system. The metric of the space $\mathbb{E}$ is given by [23]

$$\mathrm{d}l^2_{\mathrm{GTD}} = \left(E^c \frac{\partial \Psi}{\partial E^c}\right) \left(\eta_{ad} \delta^{di} \frac{\partial^2 \Psi}{\partial E^i \partial E^b}\right) \mathrm{d}E^a \mathrm{d}E^b \,, \tag{5.1}$$

where $\eta_{ad} = (-1, 1, ..., 1)$. This metric is invariant under Legendre transformations [15].

We consider an RN black hole surrounded by quintessence and immersed in a heat bath at fixed temperature, we assume no fluctuations in c ($\mathrm{d}c = 0$): This is the CE subject to (4.3). It is then convenient to use the mass (3.3) as the thermodynamic potential. We can write the matrix (5.1) in a metric form

$$\begin{aligned} \mathrm{d}l^2_{\mathrm{GTD}} &= \left(s\frac{\partial M}{\partial s} + q\frac{\partial M}{\partial q}\right)\left(-\frac{\partial^2 M}{\partial s^2}\mathrm{d}s^2 + \frac{\partial^2 M}{\partial q^2}\mathrm{d}q^2\right) \\ &= \frac{Y}{8s^{5/2}}\{[s - 3q^2 + 6c\omega(2+3\omega)s^{W_-}]\mathrm{d}s^2 + 8s^2\mathrm{d}q^2\} \end{aligned} \tag{5.2}$$

and the associated curvature scalar reads

$$\begin{aligned} R_{\mathrm{GTD}} = &-\frac{X}{Y^3}\Big[\frac{(-3q^2 + s - 18c\omega^2 s^{W_-})^2 + 18q^2[-3q^2 + s + 6c\omega(2+3\omega)s^{W_-}]}{64s^3}\Big] \\ &+ \frac{X^2}{2Y}\Big[\frac{2s + 3c\omega(10 + 21\omega + 9\omega^2)s^{W_-}}{16s^{9/2}}\Big] + \frac{3}{2Y^2}\Big\{X\,\frac{q^2 - s + 6c\omega^2(5+6\omega)s^{W_-}}{16s^{5/2}} - 1 \\ &+ X^2\,\frac{(3q^2 - s + 18c\omega^2 s^{W_-})[-5q^2 + s + 2c\omega(8 + 18\omega + 9\omega^2)s^{W_-}]}{128s^5}\Big\} \end{aligned} \tag{5.3}$$

where

$$X = \frac{C_q}{T} = \frac{8s^{5/2}}{3q^2 - s - 6c\omega(2+3\omega)s^{W_-}} \tag{5.4}$$

$$Y = sT + qA_0 = \frac{3q^2 + s + 6c\omega s^{W_-}}{4\sqrt{s}}\,. \tag{5.5}$$

It is obvious that $R_{\rm GTD}$ diverges at the points where X diverges corresponding to $C_q = \infty$. A divergence in the value of C_q announces a second order phase transition as derived in subsection 4.1.1. $R_{\rm GTD}$ diverges at $Y = 0$ too but this equation, based on the analysis made in subsections 4.1.1 and 4.1.2, represents no physical effect. This is a pathologic effect due to the fact that the metric (5.2) becomes singular at $Y = 0$.

5.2 Liu-Lu-Luo-Shao method

The second geometric method was developed later by Liu-Lu-Luo-Shao (LLLS) [18]. The method relies on the same principal as the first one in that any singularity in the curvature scalar of the associated metric signals a second order phase transition of the system. The metric of this space is simply the Hessian matrix of the Helmholtz free energy which, in the CE, is given by

$$\mathrm{d}l^2_{\rm LLLS} = -\mathrm{d}T\,\mathrm{d}s + \mathrm{d}A_0\mathrm{d}q = -\frac{\partial T}{\partial s}\mathrm{d}s^2 + \left(\frac{\partial A_0}{\partial s} - \frac{\partial T}{\partial q}\right)\mathrm{d}s\mathrm{d}q + \frac{\partial A_0}{\partial q}\mathrm{d}q^2\,. \tag{5.6}$$

Using the expressions (3.5) and (3.6), (5.6) becomes

$$\mathrm{d}l^2_{\rm LLLS} = \frac{1}{8s^{5/2}}\{[s - 3q^2 + 6c\omega(2+3\omega)s^{W_-}]\mathrm{d}s^2 + 8s^2\mathrm{d}q^2\} \qquad (\mathrm{d}l^2_{\rm GTD} = Y\,\mathrm{d}l^2_{\rm LLLS})\,. \tag{5.7}$$

This metric provides the following curvature scalar

$$R_{\rm LLLS} = \frac{[2s + 3c\omega(10 + 21\omega + 9\omega^2)s^{W_-}]}{32s^4}X^2\,. \tag{5.8}$$

Here again we clearly see that the divergence of the curvature scalar (5.8) corresponds to that of C_q, thus by the LLLS method we are too able to locate the points where the second order phase transition takes place.

6 Conclusion

We have seen that asymptotically flat RN black holes surrounded by quintessence have higher entropies than ordinary ones. For a given value of the quintessence density c, they commulate more electric charges than ordinary RN holes before they become naked singularities. For $c \ll 1$, the maximum relative cumulated charge $(q^2 - M^2)/M^2$ is proportional to $c/M^{3\omega+1}$, if $-1/3 < \omega < 0$, or to $2c/(1-2c)$, if $\omega = -1/3$.

Taking c as a thermodynamic variable, as some works do for the cosmological constant [24, 25], we have obtained the generalized Smarr formula and the first law of thermodynamics.

As one charges adiabatically an ordinary RN black hole, it cumulates mass and charge till their total values equate the radius of the horizon, which remains constant during the process. If the RN black hole is surrounded by quintessence then it will cumulate more mass and charge, however, with their totals never exceeding the radius of the horizon.

Applying the classical thermodynamic method and restricting ourselves to the CE we have obtained generalized conditions for local stability of RN black holes surrounded by quintessence. These conditions are the shifting of the known ones for ordinary RN black holes if fluctuations in T are not allowed; If T fluctuates, the same conditions apply with their upper limits constrained to lower values. We have reached the conclusion that , allowing fluctuations in (s,q,T), only black holes with low entropy, or high charge or both are stable, while black holes with relatively high entropy, or low charge or both are stable against fluctuations in (s,q) only. Between these and the other black hole states there is a behavioral change maintaining the sign of C_q.

We have completed the analysis of stability by applying the TPM and obtained an upper limit for the temperature ($T_2 = T(s_2)$) beyond which the CE is no longer stable, if fluctuations in T are not allowed. For ordinary RN black holes $T_2 = 1/(6\sqrt{3}q)$. If T fluctuates, we have obtained an upper limit for the electric potential on the horizon ($A_0(s_5)$) beyond which the CE is no longer stable. Another general result we could derive is that isolated black holes, which constitute MEs, are stable.

By the two geometrical method (GTD, LLLS) we could also determine the states corresponding to a second order phase transition.

The stability of non-asymptotically flat solutions is more involved and constitutes the matter of a subsequent work along with the case where c fluctuates.

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